

Research Article

Analytical Solution of Fractional Model of Pollution for a System Lakes

Ehtasham Ul Haq*

Department of Mathematics, COMSATS University Islamabad, Attock, Pakistan.

Keywords

System of ODE,
Variation of parameters
method,
Adomian decomposition
method,
Daftardar-Jaffari method,
Residual error.

Abstract

In this work, we will examine a fractional system of differential equations. This system is modelled for the contamination of an arrangement of lakes. We have to find the analytic results of this model by utilizing analytical method familiar as Variation of Parameters method (VPM). This technique reduces the computational work and converges to its solution by removing noise terms. These terms frequently occur in an iteration process of some techniques. To manage a fractional expression, we are coupling Riemann-Liouville fractional integral. We will also analyze our outcomes with some other familiar analytical methods like Variation of iteration technique, Adomian decomposition technique and Daftardar-Jafari technique to examine the ability of our applied approach.

1. Introduction

Pollution is causing a great damage to our environment. Pollutants, May be naturally present in the environment or they are added by human activities. Water pollution, air pollution and noise pollution are some of the forms of pollution. One of the most dangerous form of pollution is water pollution that is the reason behind the diseases like Diaphorrea, Chdera etc. To tackle this problem the first step to be taken is analytic observation of pollution. To obtain the analytic results many models are available. One of those model is "Compartment Modelling". It involves the use of differential equations, see [1,2] Given system consist of three lakes that are connected together by channel. This model contains three lakes and each lake is associated with an input that determines the amount of pollution in the respective lake.

In the history, less attention has been paid over generalization of this model. In this work the original model is generalized by fractional system of lakes. Original model was proposed by Baizar et al. [2] For the analytical solution of this model the method used is Variation of Parameters Method (VPM) [3] Analytical response of this model is given by three different inputs i.e step, impulse and sinusoidal input.

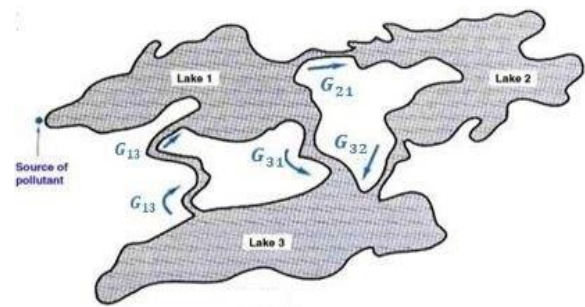


Figure 1. Arrangement of Lakes with interconnecting Channels

Figure 1. shows the arrangement of three lakes with interconnecting channels. Every lake is viewed as a vast compartment in addition the interconnecting channel as funnels between the compartments with given stream directions. The heading of stream in the pipes or funnels is represented in form of arrows as shown in Figure 1. Initially a contamination is brought into the first lake, where the rate of pollutants entering into the lake is represented by $P(t)$. The function $P(t)$ might be steady or another hand may differ with time. Our concern is to determine the rate of pollution in time t . We are occupied with knowing the level of contamination in every lake at any time t .

* Corresponding Author: Ehtasham Ul Haq
E-mail address: ahteshamkhan3801@gmail.com

Received: 17 September 2020; Revised: 11 October 2020; Accepted: 14 October 2020

We let $y_i(t)$ signify the measure of contamination in lake i whenever $t \geq 0$ where $i = 1,2,3$. By using some mixing procedure, we accept that the contamination in every lake to be consistently appropriate all through the lake. All of these lakes, we suppose that the volume of water in lake 1 is remains steady. For an instant of time, convergence of contaminant in lake 1 is given as

$$C_i(t) = \frac{y_i(t)}{V_i(t)} \tag{1}$$

At first, every lake is supposed to be devoid of any pollutant to demonstrate the dynamic conduct of arrangement of lakes, we consider that the flow rate between lake i to lake j is consider as constant. As there is no medium which is capable of spilling out from lake 2 to lake 1 i.e $G_{12} = 0$. The flow rate of escape of contaminant out of lake i to lake j at any interval of time is defined as

$$r_{ji}(t) = G_{ji}C_i(t) = \frac{G_{ji}y_i(t)}{V_i(t)} \tag{2}$$

We will see that

Rate of change of pollutant = input rate – output rate

Applying this rule to each lake gives the system of first order differential equations. In general form the aforementioned system is given as follows

$$\begin{aligned} \frac{d^\alpha y_1}{dt^\alpha} &= \frac{G_{13}}{V_3} y_3(t) - \frac{G_{31}}{V_1} y_1(t) - \frac{G_{21}}{V_1} y_1(t) + P(t) \\ \frac{d^\alpha y_2}{dt^\alpha} &= \frac{G_{21}}{V_1} y_1(t) - \frac{G_{32}}{V_2} y_2(t) \\ \frac{d^\alpha y_3}{dt^\alpha} &= \frac{G_{31}}{V_1} y_1(t) + \frac{G_{32}}{V_2} y_2(t) - \frac{G_{13}}{V_3} y_3(t) \end{aligned} \tag{3}$$

With initial conditions $y_1(0) = 0, y_2(0) = 0$ and $y_3(0) = 0$.

Here $y_1(t), y_2(t), y_3(t)$ are the un-known functions, $G_{13}, G_{31}, G_{21}, G_{32}, V_1, V_2$ and V_3 are the appropriate constants.

For the volume of every lake to stay steady, the flow rate into every lake must adjust the flow out of the lake so we acquire the following conditions:

$$\begin{aligned} \text{Lake 1: } G_{13} &= G_{31} + G_{21} \\ \text{Lake 2: } G_{21} &= G_{32} \\ \text{Lake 3: } G_{31} + G_{32} &= G_{13} \end{aligned}$$

There are the following values of the parameters of Eq. (3).

$$\begin{aligned} V_1 &= 2900mi^3, V_2 = 850mi^3, V_3 = 1180mi^3 \\ G_{21} &= 18mi^3/year, G_{32} = 18mi^3/year, \\ G_{31} &= 20mi^3/year, G_{13} = 38mi^3/year. \end{aligned}$$

The special case of this pollution model for $\alpha = 1$ that was presented by Biazar et al. [2] in 2006. In more research will be carry out in analytical and numerical techniques were utilized and exploited for this model corresponding Adomian decomposition method (ADM) and Runge Kutta method (R-K 4) by Baizar et al. [2], Variational iteration method by Baizar et al. [5], Differential transform method by Biazar et

al. [6], collocation approach by Yuzbaş et al. [7], modified differential transform technique by Benhammouda et al. [8], homotopy perturbation technique by Merdan [9] and Perturbation iteration technique by Khalid et al. [10]

In this work, by applying some analytical approach, we need to check the analytical feedback of the contamination. In analytical feedback, we get continued result which is also informational as related to numerical result. Therefore we apply Variation of parameters method (VPM) [3] to solve the model analytically. Variation of parameters method mostly used for solving the wide class of initial and boundary value problem and non-linear boundary value problems [11,12] It is worth mentioning that Ma et al. [11,12] utilized Variation of method for solving the non-homogeneous PDE's. The suggested method is free of round off error, discretization, linearization, Adomian's polynomial and utilize only initial condition, which are easier to implement and reduce the computational work.

In this work, we are utilizing Variation of parameters method (VPM) for solving the fractional model of pollution for an arrangement of lakes. We have discussed the comparison of our results with some other recognized analytical technique alike Variation of iterational method [14, 15], Adomian decomposition method (ADM) [16] and Daftardar-Jafari method (DJM) [17]

2. Prelimaniries

In this section, we have used some basic definitions and ideas with respect to fractional integral and residual error. The two most normally utilized definitions are Riemann - Liouville and Caputo. Here we are using Riemann-Liouville fractional integral for order $0 < \alpha < 1$.

2.1. Definition

The Riemann -Liouville fractional integral of order α is defined as [4]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi) d\xi, \quad t > 0 \tag{4}$$

2.2. Definition

Residual of an observed value is the difference between the approximate value of the quantity of interest and the observed value. For checking the accuracy of a function residuals are essential to be understood. The approximate solution of g is g_α .

$$R.E = \int_0^\xi |f(\xi) - T(g_\alpha)(\xi)|^2 d\xi, \tag{5}$$

The approximate solution of g over the domain 0 to ξ is considered g_α . Whenever the approximation is close to the solution, there will be smallness of the residual.

$$\left| \frac{g_\alpha(\xi) - g(\xi)}{g(\xi)} \right| \leq 1$$

3. Variation of Parameters Method

To emanate the main concept of the variation of parameters method [3,18], we assume the general Eq. (6)

$$L(w) + N(w) + R(w) = f(x) \quad a \leq x \leq b \quad (6)$$

Where L and N is a linear and non-linear operators respectively, but L has the highest order than R. w is a function of x. f(x) is a source term in the given domain [a, b]. By utilizing the VPM, we have the pursuing solution of the equation

$$w(x) = \sum_{i=0}^{n-1} \frac{C_{i+1}x^i}{i!} + \int_0^x \lambda(x, \xi)(-N(w)(\xi) - R(w)(\xi) + f(\xi))d\xi, \quad (7)$$

Where n represent the order of given differential equation and C_i where $i = 1, 2, 3, \dots$ Are unknowns.

$$\sum_{i=0}^{n-1} \frac{C_{i+1}x^i}{i!} \quad (8)$$

For homogeneous solution which is taken by

$$L(w) = 0,$$

The another part which is obtained by Eq. (6) by using VPM.

$$\int_0^x \lambda(x, \xi)(-N(w)(\xi) - R(w)(\xi) + f(\xi))d\xi, \quad (9)$$

Here $\lambda(x, \xi)$ is a Lagrange multiplier, that expel the progressive use of integrals in the iterative scheme and it depending on the order of equation. Commonly the pursue definition is utilized to find the value of the multiplier $\lambda(x, \xi)$ from

$$\lambda(x, \xi) = \sum_{i=1}^n \frac{(-1)^{i-1} \xi^{i-1} x^{n-i}}{(i-1)!(n-1)!} = \frac{(x-\xi)^{n-1}}{(n-1)!} \quad (10)$$

Here n is the order of the given differential equation and it varies for different values of n. we have the pursue cases

$$\begin{aligned} n=1, \quad \lambda(x, \xi) &= 1, \\ n=2, \quad \lambda(x, \xi) &= (x - \xi), \\ n=3, \quad \lambda(x, \xi) &= \frac{x^2}{2!} + \frac{\xi^2}{2!} - \xi x, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Therefore, we utilize the pursue iterative scheme for solving equation

$$w_{n+1} = w_0 + \int_0^x \lambda(x, \xi)(-N(w)(\xi) - R(w)(\xi) + f(\xi))d\xi, \quad (11)$$

We can get the initial guess $w_0(x)$ by using initial conditions. It is observed that the we are taking better approximation by using fixed value of initial guess in each iteration. Which have the following form for linear problem, i.e.,

$$w_{n+1} = w_0 + \int_0^x \lambda(x, \xi)(-R(w)(\xi) + f(\xi))d\xi \quad (12)$$

4. Numerical Application

Here we applied Variation of Parameters Method (VPM) on fractional model of pollution for system of lakes and also we coupled the VPM with Riemann-Liouville fractional integral, i.e

$$\begin{aligned} \frac{d^\alpha y_1}{dt^\alpha} &= y_1^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{G_{13}}{V_3} y_3^{(k)}(\xi) - \frac{G_{31}}{V_1} y_1^{(k)}(\xi) - \frac{G_{21}}{V_1} y_1^{(k)}(\xi) + p(\xi) \right) d\xi, \\ \frac{d^\alpha y_2}{dt^\alpha} &= y_2^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{G_{21}}{V_1} y_1^{(k)}(\xi) - \frac{G_{32}}{V_2} y_2^{(k)}(\xi) \right) d\xi, \\ \frac{d^\alpha y_3}{dt^\alpha} &= y_3^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{G_{31}}{V_1} y_1^{(k)}(\xi) + \frac{G_{32}}{V_2} y_2^{(k)}(\xi) - \frac{G_{13}}{V_3} y_3^{(k)}(\xi) \right) d\xi, \end{aligned} \quad (13)$$

Where $0 < \alpha < 1$ with initial conditions $y_1(0) = 0$, $y_2(0) = 0$ and $y_3(0) = 0$.

There are three different input repeses including sinusoidal, impulse and step input to guide the duration of contaminant in every lake.

4.1. Sinusoidal Input

We are using this input model, when the contaminant is introduced in the lake 1 periodically. So we take $P(t) = c + asin\omega t$, where c is the average input of concentration of contaminant, a is the amplitude of variations and ω is the frequency of variations. Taking $a = c = \omega = 1$, then Eq. (13) becomes

$$\begin{aligned} y_1^{(k+1)}(t) &= y_1^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{38}{1180} y_3^{(k)}(\xi) - \frac{38}{2900} y_1^{(k)}(\xi) + 1 + \sin(\xi) \right) d\xi, \\ y_2^{(k+1)}(t) &= y_2^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{18}{2900} y_1^{(k)}(\xi) - \frac{18}{850} y_2^{(k)}(\xi) \right) d\xi, \\ y_3^{(k+1)}(t) &= y_3^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{20}{2900} y_1^{(k)}(\xi) + \frac{18}{850} y_2^{(k)}(\xi) - \frac{38}{1180} y_3^{(k)}(\xi) \right) d\xi, \end{aligned} \quad (14)$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 0$ and $y_3(0) = 0$. They are the following results of the system for different values of alpha.

Table 1. Numerical results of Sinusoidal Input for lake 1

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.63820367	0.66232218	0.64812583	0.62051
1.0	1.13693043	1.35190705	1.41606687	1.45115
1.5	1.60814611	2.07125676	2.25796816	2.40823
2.0	2.00967829	2.74151645	3.07582877	3.37640

Table 2. Numerical results of Sinusoidal Input for lake 2

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.0011999	0.0011488	0.0010384	0.000898
1.0	0.0031019	0.0039184	0.0040684	0.004043
1.5	0.0055209	0.0081971	0.0092263	0.009940
2.0	0.0083021	0.0138148	0.0164611	0.018775

Table 3. Numerical results of Sinusoidal Input for lake 3

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.0011999	0.0012785	0.0011555	0.000999
1.0	0.0031019	0.0043660	0.0045326	0.004504
1.5	0.0055209	0.0091428	0.0102910	0.011086
2.0	0.0083021	0.0154240	0.0183818	0.011086

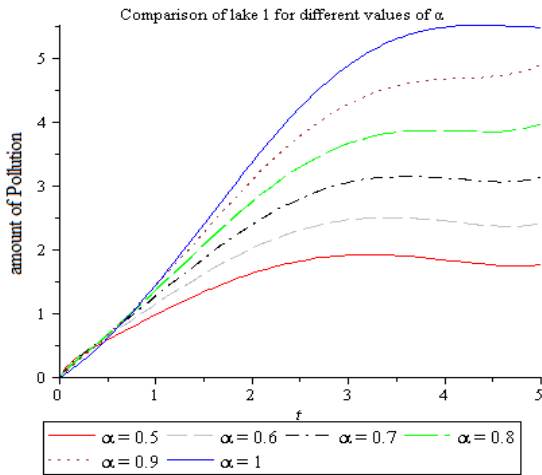


Figure 2. Represent the amount of pollution in lake 1.

We see the amount of pollution increase as we increase the values of alpha and time t. For simplicity we say that amount of pollution is directly proportional to α and time. If we take the small value of t then the lake 1 shows the behaviour sinusoidally.

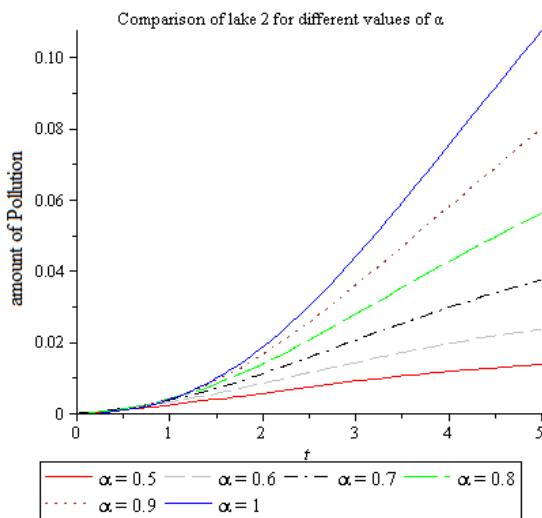


Figure 3. Represent the amount of pollution in lake 2.

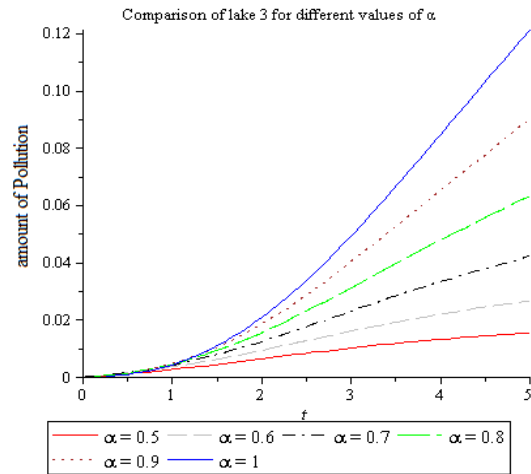


Figure 4. Represent the amount of pollution in lake 3.

The comparison of three lakes for sinusoidal input, the amount of pollution in lake 1 is high as compare to lake 2 and lake 3 because the primary source of pollutant is lake 1. From Figure 1 there is no flow between lake 2 to lake 1 so there is big reason behind the lake 2, the amount of pollution is low as compare to lake 3. The behaviour of lake 2 and lake 3 for $\alpha = 1$ that will represent the amount of pollution growing exponentially.

4.2. Impulse Input

Impulse input method is utilized, when the contaminant have been released immediately into the lake, So we take $p(t) = 100$ units. Then Eq.(13) becomes

$$\begin{aligned}
 y_1^{(k+1)}(t) &= y_1^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{38}{1180} y_3^{(k)}(\xi) - \frac{38}{2900} y_1^{(k)}(\xi) + 100 \right) d\xi, \\
 y_2^{(k+1)}(t) &= y_2^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{18}{2900} y_1^{(k)}(\xi) - \frac{18}{850} y_2^{(k)}(\xi) \right) d\xi, \\
 y_3^{(k+1)}(t) &= y_3^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{20}{2900} y_1^{(k)}(\xi) + \frac{18}{850} y_2^{(k)}(\xi) - \frac{38}{1180} y_3^{(k)}(\xi) \right) d\xi,
 \end{aligned}
 \tag{15}$$

With initial conditions $y_1(0) = 0, y_2(0) = 0$ and $y_3(0) = 0$ They are the following results of the system for different values of alpha.

Table 4. Numerical results of Impulse Input for lake 1

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	50.841435	52.961462	51.934695	49.83702
1.0	76.892637	91.945293	96.612434	99.35139
1.5	97.897925	126.84707	138.75266	148.5480
2.0	116.16391	159.29033	179.25161	197.4318

Table 5. Numerical results of Impulse Input for lake 2

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.1031943	0.0986443	0.0891418	0.077142
1.0	0.2360513	0.2974220	0.3086545	0.306798
1.5	0.3826340	0.5662336	0.6369228	0.686307
2.0	0.5387271	0.8931089	1.0633907	1.213009

Table 6. Numerical results of Impulse Input for lake 3

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.1148860	0.1097915	0.0991980	0.085829
1.0	0.2630651	0.3314546	0.3439342	0.341810
1.5	0.4267794	0.6317567	0.7106347	0.765679
2.0	0.6013221	0.9975458	1.1879437	1.355174

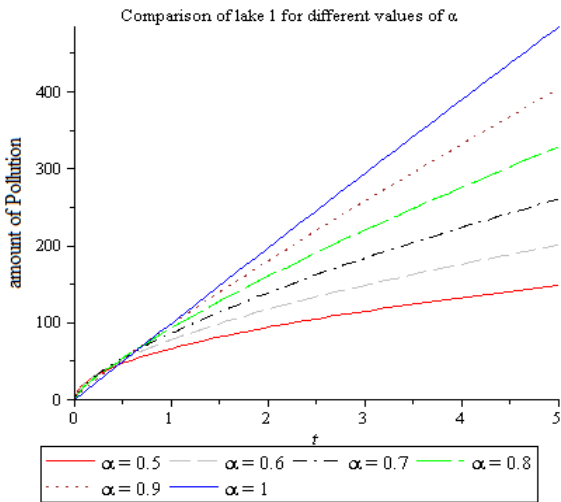


Figure 5. Represent the amount of pollution in lake 1.

We see the amount of pollution increase as we increase the values of alpha and time t. The behaviour of lake 1 for $\alpha = 1$ that will represent the amount of pollution increase with constant concentration.

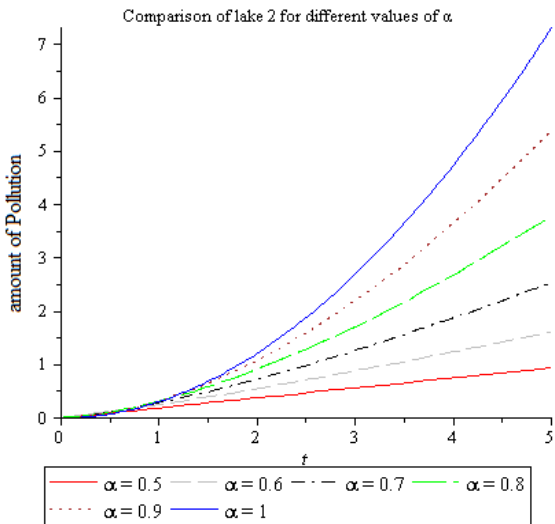


Figure 6. Represent the amount of pollution in lake 2.

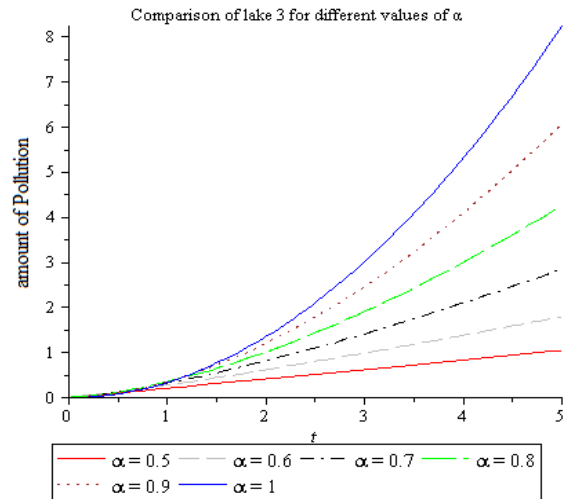


Figure 7. Represent the amount of pollution in lake 3.

The comparison of three lakes for impulse input, the amount of pollution in lake 1 is high as compare to lake 2 and lake 3 because the primary source of pollutant is lake 1. From Figure 1 there is no flow between lake 2 to lake 1 so there is big reason behind the lake 2, the amount of pollution is low as compare to lake 3. The behaviour of lake 2 and lake 3 For $\alpha = 1$ that will represent the amount of pollution growing exponentially

4.3. Step Input

Step input method is used, when the pollutant introduced into the first lake with linear concentration, then we take $p(t) = ct$, here c is a positive constant. If we choose $c = 100$ Then Eq. (13) becomes

$$\begin{aligned}
 y_1^{(k+1)}(t) &= y_1^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{38}{1180} y_3^{(k)}(\xi) - \frac{38}{2900} y_1^{(k)}(\xi) + 100\xi \right) d\xi \\
 y_2^{(k+1)}(t) &= y_2^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{18}{2900} y_1^{(k)}(\xi) - \frac{18}{850} y_2^{(k)}(\xi) \right) d\xi \\
 y_3^{(k+1)}(t) &= y_3^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \left(\frac{20}{2900} y_1^{(k)}(\xi) + \frac{18}{850} y_2^{(k)}(\xi) - \frac{38}{1180} y_3^{(k)}(\xi) \right) d\xi
 \end{aligned}
 \tag{16}$$

With initial conditions $y_1(0) = 0, y_2(0) = 0$ and $y_3(0) = 0$. They are the following results of the system for different values of alpha.

Table 7. Numerical results of Step Input for lake 1

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	13.262336	13.556041	13.152377	12.472803
1.0	40.148121	47.115966	48.985561	49.783250
1.5	76.722907	97.588547	105.63269	111.77123
2.0	121.45184	163.53337	182.12566	198.27912

Table 8. Numerical results of Impulse Input for lake 2

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.0170160	0.0164418	0.0148874	0.0128756
1.0	0.0779485	0.0992942	0.1032494	0.1025617
1.5	0.1897320	0.2839369	0.3200456	0.3446497
2.0	0.3565101	0.5978875	0.7134446	0.8134014

Table 9. Numerical results of Impulse Input for lake 3

t	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.0	0.0	0.0	0.0	0.0
0.5	0.0189326	0.0182913	0.0165603	0.0143206
1.0	0.0867904	0.1105656	0.1149645	0.1141882
1.5	0.2113766	0.3164330	0.3566949	0.3841126
2.0	0.3973827	0.6668395	0.7958724	0.9074721

Now the graphical representation of the above model is shown below

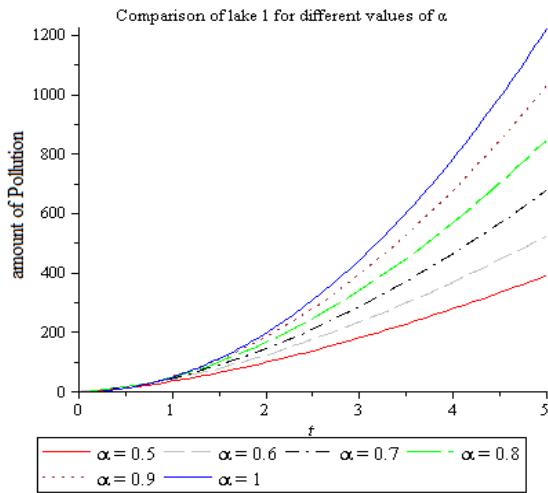


Figure 8. Represent the amount of pollution in lake 1.

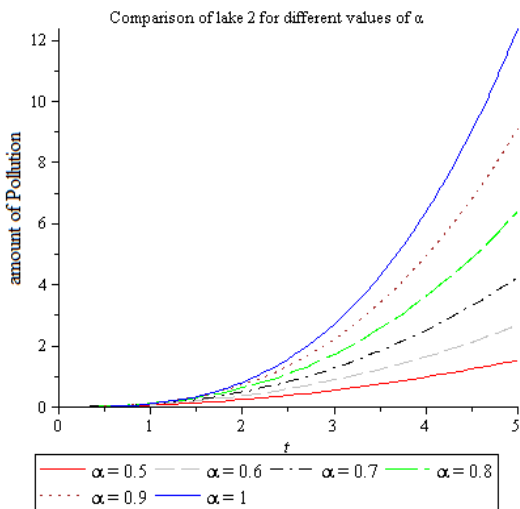


Figure 9. Represent the amount of pollution in lake 2.

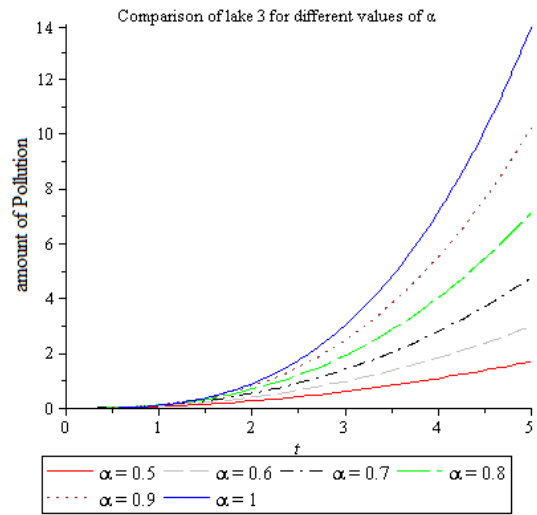


Figure 10. Represent the amount of pollution in lake 3.

The comparison of three lakes for step input, the amount of pollution in lake 1 is high and growing rapidly as compare to lake 2 and lake 3 because the primary source of pollutant is lake 1. From Figure 1 there is no flow between lake 2 to lake 1 so there is big reason behind the lake 2, the amount of pollution is low as compare to lake 3. The behaviour of lake1, 2 and lake 3 For $\alpha = 1$ that will represent the amount of pollution growing exponentially.

5. Special Case

Now we are comparing our techniques for $\alpha = 1$ with Variational iteration method (VIM) and Adomian decomposition method (ADM) by fixing the percentage of the pollution from lake 1 and lake 3.

- The first lake loses 65% of its pollution to the 3rd lake and is gaining 10% of the Pollution from the 3rd lake.
- The 3rd lake loses 10% of its pollution to the first lake and is gaining 65% of the Pollution from the first lake yearly,

We are comparing our techniques by finding the residual error. So the residual error of lake 1, 2 and 3 are shown below.

Table 10. Residual Error of Sinusoidal Input.

Lakes	VPM	VIM	ADM	DJM
Lake 1	1.73×10^{-1}	1.73×10^{-1}	3.45×10^1	1.58×10^2
Lake 2	1.11×10^{-5}	1.11×10^{-5}	2.43×10^{-5}	5.4×10^{-3}
Lake 3	1.7×10^{-1}	1.7×10^{-1}	5.41×10^{-1}	6.49×10^1

Table 11. Residual Error of Step Input.

Lakes	VPM	VIM	ADM	DJM
Lake 1	1.73×10^2	1.73×10^2	1.44×10^5	1.21×10^5
Lake 2	1.11×10^{-2}	1.11×10^{-2}	4.31×10^{-2}	1.99
Lake 3	1.7×10^2	1.7×10^2	9.31×10^2	2.24×10^4

Table 12. Residual Error of Impulse Input.

Lakes	VPM	VIM	ADM	DJM
Lake 1	8.9×10^2	8.9×10^2	1.15×10^5	2.74×10^5
Lake 2	5.72×10^{-2}	5.72×10^{-2}	1.02×10^{-1}	4.89
Lake 3	8.74×10^2	8.74×10^2	2.32×10^3	5.47×10^4

In above table we see that VPM and VIM give the best residual error as compare to ADM and DJM. For the solution procedure VPM is more efficient and simplest technique as compare to VIM.

6. Conclusions

In this paper, we are using variation of parameters method for solving fractional model of pollution for system of lakes. The behaviour of arrangement of lakes for different values of alpha are shown in Figure (2-10). We are using different inputs are used to monitor the amount of pollution in each lake. Above all figures represent the amount of pollution for different values of alpha, when the pollutant introduced in 1st lake with periodic, constant and linear concentration. We have concluded that the amount of pollution increase as well as increase the values of alpha and time t, in each input we see that the amount of pollutant in lake 1 is high as compare to lake 2 and 3, because the primary source of pollutant is lake 1. The comparison shows that VPM and VIM gives the least residual error as compare to ADM and DJM, but the solution process of VPM is much easier as compare to VIM, ADM and DJM. We investigate that our proposed technique is appropriate for such physical problem as its gives solution in less number of iterations. It is worth mentioning that the method is capable of reducing the computational work as compare to other analytical and numerical methods.

Conflict of Interest Statement

The authors declare no conflict of interest.

References

- [1] J. Agurre and D. Tully, Lake pollution Model, May 21, 1999.
- [2] J. Biazar, L. Farrokhi and M.R. Islam, Modeling the pollution of a system of lakes, Journal of Applied Mathematics and Computation 178 (2006) 423–430.
- [3] S.T. Mohyud-Din, M.A. Noor and A. Waheed, Variation of parameters method for initial and boundary value problem, World Applied Sciences Journal, 11 (2010) 622–639.
- [4] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Inter Science Publication. John Wiley & Sons Inc., New York, 1993.
- [5] J. Biazar, M. Shahbala and H. Ebrahimi, VIM for solving the pollution problem of a system of lakes equations, Journal of Control Science and Engineering, Article ID 829152 (2010), 6 pages.
- [6] J. Biazar and T. Rahimi, Differential transform method for the solution of the lake problem, Journal of Natural Science and Sustainable Technology 6 (2013) 103–113.

- [7] S. Yuzbaş, N. Şahin and M. Sezer, A collocation approach to solving The model of pollution for a system of lakes, Journal of Mathematical and Computer Modelling 55 (2012) 330–341.
- [8] B. Benhammouda, H. Vazquez-Leal and L. Hernandez-Martinez, Modified differential transform method for solving the model of pollution for a system of lakes, Discrete Dynamics in nature and Society, Article ID 645726 (2014) 12.
- [9] M. Merdan, Homotopy perturbation method for solving modeling the pollution of a system of lakes, Suleyman Demirel University Journal of Science (E-Journal) 4 (2009) 99–111.
- [10] M. Khalid, M. Sultana, F. Zaidi and F. Sami Khan, Solving polluted lakes system by using Perturbation-Iteration Method, International Journal of Computer Application 114 (2015) 1–7.
- [11] W. X. Ma, Y. You, Solving the Korteweg-de Varies equation by its bilinear form: Wronskian solutions. Transactions of the American Mathematical Society 357 2004 395–406.
- [12] W. X. Ma, Y. You, Rational solutions of the Toda lattice equation in Casoratian form. Chaos, Solitons and Fractals 22 (2004) 395–406.
- [13] M. A. Noor, S.T. Mohyud-Din, A. Waheed, Variation of parameters method for solving fifth-order boundary value problems, Journal of Applied Mathematics and information Science 2 (2008) 145–141.
- [14] M. A. Abdou, A. A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, Journal of Computational and Applied Mathematics 181 (2005) 245–251.
- [15] M. A. Abdou, S. T. Mohyud-Din, Variational iteration method for solving higher order non-linear boundary value problems using He's polynomials, International Journal of non-linear Science and Numerical Simulation 9 (2008) 141–157.
- [16] A. M. Wazwaz, The decomposition method for solving higher dimensional initial boundary value problems of variable coefficients, International Journal of Computational Mathematics 76 (1999) 159–172.
- [17] V. Daftardar-Gejji, H. Jafari, An iterative method for solving non-linear functional equations, Journal of Mathematical Analysis and Applications 316 (2006) 753–763.
- [18] E. U. Haq, A. Saeed Khan, M. Ali, An efficient approach for solving the linear and nonlinear integro differential equations, Computational Research Progress in Applied Sciences and Engineering 6 (2020) 104–107.